

INTEGER SYMMETRIC MATRICES OF SMALL SPECTRAL RADIUS AND SMALL MAHLER MEASURE

JAMES MCKEE AND CHRIS SMYTH

ABSTRACT. In a previous paper we completely described *cyclotomic* matrices—integer symmetric matrices of spectral radius at most 2. In this paper we find all minimal non-cyclotomic matrices. As a consequence, we are able to determine all integer symmetric matrices of spectral radius at most 2.019, and to determine all integer symmetric matrices whose Mahler measure is at most 1.3. In particular we solve the strong version of Lehmer’s problem for integer symmetric matrices: all noncyclotomic matrices have Mahler measure at least ‘Lehmer’s number’ $\lambda_0 = 1.17628\dots$.

1. STATEMENT OF RESULTS

For a monic polynomial $g(x)$ with integer coefficients and degree d , define $z^d g(z + 1/z)$, a monic reciprocal polynomial of degree $2d$, to be its *associated reciprocal polynomial*. If $g(z)$ has all its roots real and in the interval $[-2, 2]$, then the roots of its associated reciprocal polynomial are all of modulus 1, and, by a theorem of Kronecker [13], it is a cyclotomic polynomial. If A is a d -by- d symmetric matrix with integer entries, then all the roots of its characteristic polynomial are real algebraic integers, and we denote by $R_A(z)$ its associated reciprocal polynomial. If A has spectral radius at most 2, so that $R_A(z)$ is cyclotomic, then we say that A is a *cyclotomic matrix*. In a previous paper [15] we completely described cyclotomic matrices.

We are interested in the spectrum of values taken by two different functions of integer symmetric matrices A . The first is the spectral radius. In this paper we find all integer symmetric matrices whose spectral radius is less than 2.019.

The second function we are interested in is the *Mahler measure* of A , defined to be the Mahler measure of $R_A(z)$, namely the product of the absolute values of all roots of $R_A(z)$ that have modulus greater than 1. The Mahler measure of A equals 1 precisely when A is a cyclotomic matrix.

Is there a (monic) polynomial with integer coefficients that has Mahler measure λ greater than 1 but such that every polynomial with integer coefficients has Mahler measure either 1 or at least λ ? This is Lehmer’s famous problem. The smallest known such Mahler measure greater than 1 is

$$\lambda_0 = 1.176280818\dots, \tag{1.1}$$

the larger real root of the polynomial

$$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

2000 *Mathematics Subject Classification.* 11R06.

Is there a Mahler measure in the open interval $(1, \lambda_0)$?

Restricting to certain classes of polynomials, the analogue of Lehmer's problem may be more accessible. For example, Borwein, Dobrowolski and Mossinghoff [2] settled the problem for those polynomials that have only odd coefficients (the Mahler measure is either 1 or at least $1.4953\dots$), and Dobrowolski [9] showed that if A is an integer symmetric matrix, then $R_A(z)$ has Mahler measure either 1 or at least

$$\lambda_1 = 1.043\dots$$

In this paper (Corollary 2), we use a completely different approach to strengthen Dobrowolski's result and to show that the Mahler measure of an integer symmetric matrix is either 1 or at least λ_0 given by (1.1). This constant is best possible here, as there are integer symmetric matrices that have Mahler measure equal to λ_0 , the smallest being

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Actually, these are the adjacency matrices of the charged signed graphs $5u$ and $5N$ of Section 6. In fact we go further: in Theorem 4 we find all integer symmetric matrices having Mahler measure less than 1.3.

Adjacency matrices of graphs form a proper subset of the set of all integer symmetric matrices: the entries are either 0 or 1, with zeros on the diagonal. For these special matrices, the analogue of Lehmer's problem was solved in [14, Corollary 10.1], based on work of Brouwer and Neumaier [4] and Cvetković, Doob, and Gutman [7]. A key ingredient in the current paper is an extension of this work on graphs to cover signed graphs (allowing -1 as an off-diagonal matrix entry) and charged signed graphs (allowing ± 1 entries on the diagonal). Matrices having any entry of modulus 2 or more are easily disposed of.

We now state our main theorem, the one from which our results on the spectrum of the spectral radius and Mahler measure of integer symmetric matrices are derived. The terms used in its statement are defined in the next section.

Theorem 1. *Up to equivalence, the minimal noncyclotomic integer symmetric matrices are those catalogued below in Section 6. There are 4 infinite families and 125 sporadic examples.*

This result generalises a theorem of Cvetković, Doob, and Gutman [7], who found all 18 minimal noncyclotomic graphs (i.e., restricting to integer symmetric matrices with $\{0, 1\}$ entries and zeros on the diagonal). See also [8, Theorem 2.3]. These 18 graphs are precisely the graphs (i.e., the charged signed graphs without charges or negatively signed edges) that appear in our catalogue: $4a$, $4j$, $4B$, $5a$, $5E$, $6c$, $6d$, $6h$, $7a$, $7b$, $8a$, $8b$, $8c$, $9a$, $9b$, $9d$, $10a$, $10f$.

By an interlacing argument, we immediately obtain the following corollary.

Corollary 2. *If A is an integer symmetric matrix, then the Mahler measure of A is either 1 or at least λ_0 given by (1.1).*

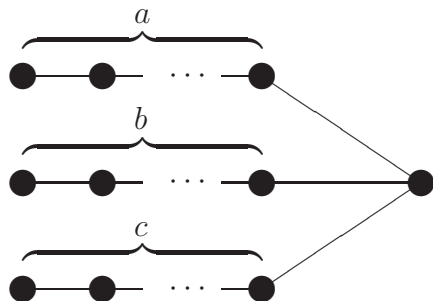
Furthermore, if A is indecomposable and has Mahler measure equal to λ_0 then it is equivalent to the adjacency matrix of one of the charged signed graphs 5u, 5N or 10f.

For the spectral radius, this implies that if an integer symmetric matrix is not cyclotomic then its spectral radius is at least $\sqrt{\lambda_0} + 1/\sqrt{\lambda_0} = 2.00659\dots$. However, we can build on Theorem 1 to obtain the following stronger results.

Theorem 3. *Up to equivalence, the indecomposable integer symmetric matrices having spectral radius less than 2.019 are either cyclotomic or have spectral radius equal to one of the ten values given in Table 1. The matrices having each such spectral radius are also given in this table.*

#	Maximum modulus of eigenvalues	Charged signed graph having corresponding spectral radius
1	2.00659	$10f = T_{1,2,6}$
2	2.00960	$10e$
3	2.01076	$11c = T_{1,2,7}$
4	2.01348	$10d, 12b = T_{1,2,8}$
5	2.01532	$9d = T_{1,3,4}, 10g, 11a, 11b, 13a = T_{1,2,9}$
6	2.01658	$10h, 14a = T_{1,2,10}$
7	2.01746	$15a = T_{1,2,11}$
8	2.01809	$16a = T_{1,2,12}$
9	2.01854	$17a = T_{1,2,13}$
10	2.01887	$12a, 18a = T_{1,2,14}$

TABLE 1. The noncyclotomic connected charged signed graphs whose eigenvalues are at most 2.019 in modulus. The graphs are drawn in Sections 6 and 7.

Figure 1: The tree $T_{a,b,c}$.

The choice of 2.019 for our bound was governed by the fact that as $n \rightarrow \infty$ the spectral radius of the graph $T_{1,2,n}$ (Figure 1) tends to $\sqrt{\theta_0} + 1/\sqrt{\theta_0} = 2.019800887\dots$. Here θ_0 is the real root of $x^3 - x - 1$. Thus if our upper bound were to be above 2.019800887... it would include infinitely many integer symmetric matrices. For graphs this was done by Cvetković, Doob, and Gutman [7] and Brouwer and Neumaier [4], who in fact found all graphs of spectral radius less than $\sqrt{2 + \sqrt{5}} = 2.058171027\dots$. See also [8, Theorem 2.4]. Furthermore Shearer [17] showed that the set of all spectral radii of graphs is dense in $(\sqrt{2 + \sqrt{5}}, \infty)$. It would be nice to extend the analysis for $(2, \sqrt{2 + \sqrt{5}})$ to general integer symmetric matrices.

For the Mahler measure, we have a corresponding result.

Theorem 4. *Up to equivalence, the indecomposable integer symmetric matrices having Mahler measure less than 1.3 are either cyclotomic or have Mahler measure equal to one of the sixteen values given in Table 2. The matrices having each such Mahler measure are also given in this table.*

Concerning the choice of 1.3 for the bound, we first note that for a finite list of matrices we must have the bound less than $\theta_0 = 1.324717957\dots$, the limit as $n \rightarrow \infty$ of the Mahler measures of the sequence of graphs $T_{1,2,n}$. Also, Boyd in his 1977 table of small Salem numbers [3] (later slightly extended both by Boyd himself and by Mossinghoff) used the bound 1.3. See also [18]. And all but two of the reciprocal polynomials of the matrices in Table 2 are (apart from possible cyclotomic factors) minimal polynomials of Salem numbers: the exceptions are the (reciprocal polynomials of the) charged signed graphs 10c and 11d. For the extended table see [16]. Of the 47 Salem numbers, our result shows that only 14 of them are Mahler measures of integer symmetric matrices.

In [14] we in fact showed that from [4] and [7] mentioned above one could describe all graphs having Mahler measure less than $\frac{1}{2}(1 + \sqrt{5}) = 1.61803\dots$. Again, it would be nice to improve Theorem 4 by increasing the bound 1.3 up to this value.

In [9] it was shown, contrary to a conjecture of Estes and Guralnick [10], that there are infinitely many totally real algebraic integers whose minimal polynomial is not the

#	Mahler measure	Charged signed graph having corresponding Mahler measure
1	1.17628	$10f = T_{1,2,6}, 5N, 5u$
2	1.18837	$9e$
3	1.20003	$8d$
4	1.21639	$5M, 5p, 7c, 10e$
5	1.21972	$9g$
6	1.23039	$11c = T_{1,2,7}, 5L, 6v$
7	1.23632	$8e$
8	1.24073	$6m$
9	1.25364	$10c$
10	1.25622	$9h$
11	1.26123	$5F, 5K, 6t, 6u, 7d, 10d, 12b = T_{1,2,8}$
12	1.26730	$8h, 9i, 10i$
13	1.28064	$4I, 5o, 5x, 6q, 6r, 6s, 8f, 9d, 10g, 11a, 11b, 13a = T_{1,2,9}$
14	1.28929	$11d$
15	1.29349	$5J, 5O, 7e, 7f, 7g, 8g, 9j, 10h, 14a = T_{1,2,10}$
16	1.29568	$9f$

TABLE 2. The noncyclotomic connected charged signed graphs whose Mahler measure is less than 1.3. The graphs are drawn in Sections 6 and 8.

characteristic polynomial of an integer symmetric matrix. The counterexamples in [9] were all cyclotomic: the conjugates of the real algebraic integers being all in the interval $[-2, 2]$. Comparing our Table 2 with the tables of small Salem numbers in [3] and [16], we find some noncyclotomic counterexamples to the conjecture of Estes and Guralnick: for example, if $x^7 - 8x^5 + 19x^3 - 12x + 1$ were the characteristic polynomial of an integer symmetric matrix A , then $R_A(z)$ would be $z^{14} - z^{12} + z^7 - z^2 + 1$, and the Mahler measure of A would be $1.20261\dots$, but this does not appear in Table 2.

2. SOME DEFINITIONS

A *permutation* of a d -by- d matrix $A = (a_{ij})$ is a matrix $P^T A P$, where P is a d -by- d permutation matrix. More generally, a *signed permutation* of $A = (a_{ij})$ is a matrix $P^T A P$, where P is a d -by- d *signed permutation matrix*, that is an element of the group $O_d(\mathbb{Z})$ of orthogonal matrices with integer entries (which must be 0 or ± 1). If such a P is a diagonal matrix, we obtain a *switching* of A . In other words, to perform a switching we change the signs of some subset of the rows, and then change the signs of the same subset of the columns. Note that diagonal entries are unchanged by a switching. In particular, *switching a vertex* means changing the signs of all edges incident at that vertex.

Two d -by- d integer symmetric matrices A and B will be called *equivalent* if $B = \pm P^T A P$ for some $P \in O_d(\mathbb{Z})$. Since a signed permutation matrix is a product of a permutation

matrix and a diagonal matrix, we see then that one can be transformed to the other by a permutation and a switching, and then possibly a change of sign of all the entries. Equivalent matrices have the same spectral radius and the same Mahler measure. Applying a permutation or a switching does not change the eigenvalues; changing the signs of all entries changes the signs of all eigenvalues. If the matrices are restricted to being the adjacency matrices of signed graphs, then our equivalence classes almost correspond to the ‘signed switching classes’ of Cameron et al. [6], except that we allow a change of sign of all the edges: our class is the union of one or two signed switching classes.

Any d -by- d integer symmetric matrix $A = (a_{ij})$ can be viewed as the adjacency matrix of a signed graph with charges: take d vertices, and for $i \neq j$ put $|a_{ij}|$ edges with the same sign as a_{ij} joining vertex i and vertex j . At the i th vertex place a charge of a_{ii} . The matrix A is called *indecomposable* if this graph is connected. If A is not indecomposable, then it is *decomposable*: this implies that some permutation of A is in block-diagonal form, with more than one block, and the eigenvalues of A are obtained by pooling the eigenvalues of its blocks. The spectral radius of A is clearly the maximum spectral radius of its blocks, while the Mahler measure of A is the product of the Mahler measures of its blocks. For our purposes it is therefore sufficient to consider indecomposable matrices.

Whenever we refer to a *subgraph*, we shall mean one induced by a subset of its vertices. A *submatrix* of a square matrix A is one that is obtained from A by deleting a subset of the rows, and deleting the same subset of the columns. With the graphical interpretation, a submatrix is the adjacency matrix of a subgraph.

Complementary to our definition of a cyclotomic matrix, we say that an integer symmetric matrix is *noncyclotomic* if its spectral radius is greater than 2. A d -by- d integer symmetric matrix is *minimal noncyclotomic* if it is noncyclotomic and every $(d-1)$ -by- $(d-1)$ submatrix is cyclotomic. A minimal noncyclotomic matrix is necessarily indecomposable.

We shall need the interlacing theorem of Cauchy (see [5, Théorème I, p.187], quoted in [1, pp.59, 78]; see also [11] for a short proof):

Lemma 5. *Let A be a d -by- d integer symmetric matrix, with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_d$. If B is any $(d-1)$ -by- $(d-1)$ principal submatrix of A , with eigenvalues $\mu_1 \leq \mu_2 \leq \dots \leq \mu_{d-1}$, then the eigenvalues of A and B interlace:*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \dots \leq \lambda_{d-1} \leq \mu_{d-1} \leq \lambda_d.$$

By interlacing, any noncyclotomic integer symmetric matrix A contains a minimal noncyclotomic submatrix M (perhaps more than one). Moreover, interlacing shows that the Mahler measure of A is at least that of M . We see that Corollary 2 follows from Theorem 1.

All of the terms in the statement of Theorem 1 have now been defined. To prove the theorem, we shall need a good description of all *maximal cyclotomic* indecomposable integer symmetric matrices A , i.e., those indecomposable cyclotomic integer symmetric A such that if A is a submatrix of B and B is cyclotomic and indecomposable then $A = B$.

3. MAXIMAL CYCLOTOMIC INTEGER SYMMETRIC MATRICES

We recall from [15] the maximal indecomposable cyclotomic integer symmetric matrices (up to equivalence). There are seven sporadic examples, $S_1, S_2, S_7, S_8, S'_8, S_{14}, S_{16}$, and three infinite families $T_{2k}, C_{2k}^{++}, C_{2k}^{+-}$.

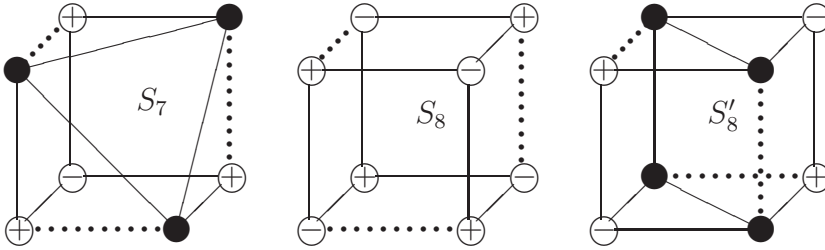
The first two sporadic examples are

$$S_1 = (2),$$

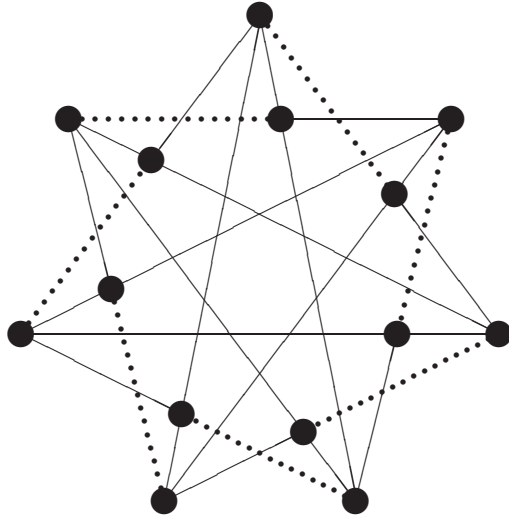
$$S_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

The remaining examples have all entries in $\{-1, 0, 1\}$, so are conveniently (and more compactly) represented by charged signed graphs, with the following conventions: (i) positive edges are represented by solid lines —; (ii) negative edges are represented by dotted lines; (iii) neutral vertices (0 on the leading diagonal) are represented by a solid disc ●; (iv) vertices with a positive charge (+1 on the diagonal) are represented by \oplus ; (v) vertices with a negative charge (−1 on the diagonal) are represented by \ominus . They are shown in the pictures below, taken from [15].

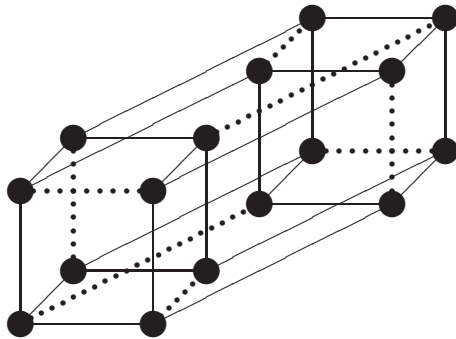
The three sporadic maximal cyclotomic charged signed graphs:



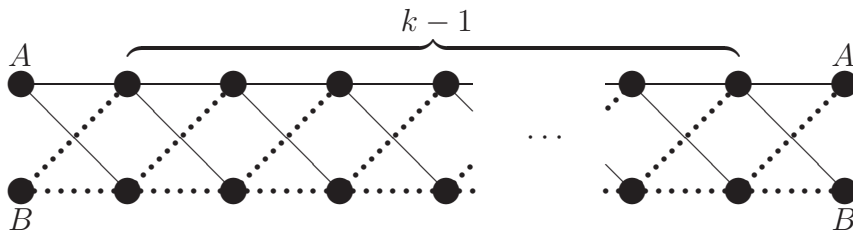
The 14-vertex sporadic maximal cyclotomic signed graph S_{14} :



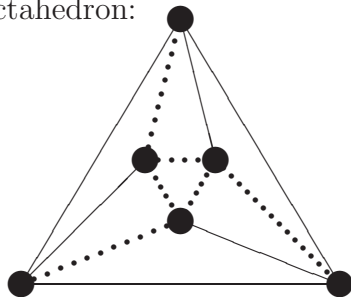
The hypercube sporadic maximal cyclotomic signed graph S_{16} :



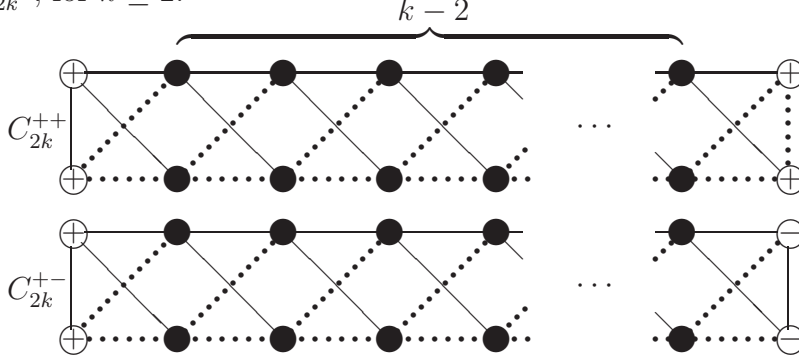
The family T_{2k} of $2k$ -vertex maximal cyclotomic toral tessellations, for $k \geq 3$ (the two copies of vertices A and B should be identified):



T_6 is a signed octahedron:



The families of $2k$ -vertex cyclotomic cylindrical tessellations C_{2k}^{++} and C_{2k}^{+-} , for $k \geq 2$:



C_4^{++} and C_4^{+-} are tetrahedra:



4. REMARKS CONCERNING T_{2k} , C_{2k}^{++} , C_{2k}^{+-} AND THEIR SUBGRAPHS

4.1. Definitions. We say that a signed graph G has a *profile* if its vertex set can be partitioned into a sequence of $k \geq 3$ subsets V_1, \dots, V_k so that either

- two vertices are adjacent if and only if for some i one belongs to V_i and the other to V_{i+1}
- or
- two vertices are adjacent if and only if for some i one belongs to V_i and the other to V_{i+1} or one belongs to V_k and the other to V_1 .

In the latter case we say that the profile is *cycling*. We call the subsets V_i the *columns* of the profile. For our application we are interested only in profiles where the V_i contain one or two vertices. In particular, this applies to T_{2k} , which has a cycling profile, and to its connected subgraphs. For a vertex v in a 2-vertex column, the other vertex in that column will be denoted \bar{v} , the *conjugate* of v .

It will be convenient to extend these definitions to charged signed graphs, insisting further that each V_i contains only neutral vertices or only charged vertices all with the same charge, and relaxing the adjacency rule to read that xy is an edge in G if and only if *either* x and y are in adjacent columns *or* are charged vertices in the same column. With this definition, both C_{2k}^{++} and C_{2k}^{+-} have a profile.

If G has a profile, then we define its *rank* to be the number of columns in the profile – this may depend on the profile, but see Lemma 6 below.

In any graph G we define, following [12], a *chordless path* or *chordless cycle* to be a path or cycle P with the property that if two vertices of P are adjacent in G then they are adjacent in P . We call the maximum number of vertices, taken over all chordless paths and chordless cycles of G , to be the (*path*) *rank* of G . As we shall show, the two definitions of rank coincide, except for small graphs.


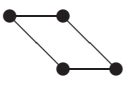
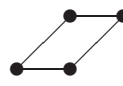
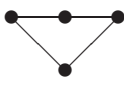
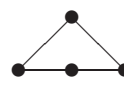
4.2. The subgraphs of T_{2k} , C_{2k}^{++} and C_{2k}^{+-} . As drawn above, the $2k$ vertices of T_{2k} or C_{2k}^{++} or C_{2k}^{+-} have a profile of rank k , with each column comprising two mutually conjugate vertices. Now let G be a connected subgraph of one of T_{2k} , C_{2k}^{++} , or C_{2k}^{+-} , drawn as above. Then G inherits a profile from one of these graphs.

Lemma 6. *Let G be equivalent to a connected subgraph of one of T_{2k} , C_{2k}^{++} , and C_{2k}^{+-} . If G has path rank at least 5 then this equals its profile rank, and its columns are uniquely determined. Moreover their order is determined up to reversal or cycling.*

We remark that the lemma is false if ‘5’ is replaced by ‘4’: the graph T_6 has path rank 4, but profile rank 3; the graph T_8 has path/profile rank 4, but the columns of its profile are not uniquely determined: if $\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}$ is a profile for T_8 , then so is $\{1, 7\}, \{2, 4\}, \{3, 5\}, \{6, 8\}$. Note also that the profile does not always determine G up to equivalence, even for high rank: for signed cycles of fixed even length there are two equivalence classes, indistinguishable by their profiles.

Proof. We start with a chordless path or cycle P of maximal number of vertices r . Because $r \geq 5$, no two of these vertices can be in the same column (this is not necessarily true for $r \leq 4$), and each column of G contains exactly one vertex in P . This shows that the profile rank equals the path rank. The columns of the profile of P inherited from that of G are singletons. We then add vertices to these columns, to complete the profile of G . Because $r \geq 5$ the column to which a new vertex can be added is completely determined by the vertices it is adjacent to in G . The last sentence of the Lemma is clear. \square

Proposition 7. (i) *Let H be a signed graph of rank at least 5 that has, for some k , the same underlying graph as a subgraph of T_{2k} , drawn as above. Then H is equivalent to a subgraph G of T_{2k} if and only if*

- the hourglass 4-cycles (with underlying graph ) all have an even number of positive edges;
- the parallelogram 4-cycles (, ) all have an odd number of positive edges;
- the triangular 4-cycles (, ) all have an odd number of positive edges.

(ii) *Let H be a charged signed graph of rank at least 5 that has, for some k , the same underlying graph as a subgraph of C_{2k}^{++} or C_{2k}^{+-} , drawn as above. Then H is equivalent to a subgraph G of C_{2k}^{++} or C_{2k}^{+-} if and only if*

- the hourglass 4-cycles all have an even number of positive edges;
- the parallelogram 4-cycles all have an odd number of positive edges;

- the triangular 4-cycles all have an odd number of positive edges;
- the triangles containing two charged vertices in the subgraph have the property that if the charges are positive (respectively negative) then the triangle has an even number of positive (resp. negative) edges.

Note that we are not claiming that the charged signed subgraph G found has the same underlying drawn graph as the charged signed graph H that we started with.

Proof. We first show that the conditions given in the Proposition are necessary. Since H has rank at least 5, its profile is, by Lemma 6, uniquely determined, and thus each 4-cycle is specified by our standard drawing of T_{2k} as being either

- an hourglass
- or
- a parallelogram 4-cycle or a triangular 4-cycle.

(If conjugate vertices are interchanged in the drawing, parallelogram 4-cycles can become triangular 4-cycles, and vice versa.) Since equivalence preserves the parity of the number of positive edges on an even cycle, we have necessity.

We now prove sufficiency. We assume that the given conditions hold, and prove that they are sufficient: that our given graph is then equivalent to a subgraph of T_{2k} , C_{2k}^{++} or C_{2k}^{+-} . To do this, we need to embed a graph equivalent to H into one of T_{2k} , C_{2k}^{++} or C_{2k}^{+-} so that the resultant embedding G inherits its edge and vertex signs from the graph it is embedded into.

- (i) Suppose that we have a signed graph H of rank at least 5 that shares the same underlying graph as a subgraph of T_{2k} , drawn as above, and satisfies the three 4-cycle conditions above.

Take a maximum-length chordless path or cycle P in H , of rank k' say. By switching we can arrange that P either has

- (a) has all edges positive;
- (b) exactly one negative edge.

In case (b) we can further assume that there is in fact no choice of P that has, after switching, all edges positive. Put $k = k'$ if P is a cycle, and $k = k' + 1$ if not. In case (a) we embed P in the top row of T_{2k} . In case (b), P is a cycle, and we can assume that $P = v_1 v_2 \dots v_k$, with only edge $v_k v_1$ negative. We then embed P into T_{2k} so that v_1, \dots, v_{k-1} are on the top row, and the final vertex v_k is one column further along, on the bottom row. Then the signs of P and T_{2k} are consistent, the edge of negative slope having positive sign, and vice versa. Note too that in this latter case the vertex \bar{v}_k cannot be present in H . For if it were, then the edges joining it to v_1 and v_{k-1} could not be of the same sign (or we could by switching make the top row all positive, and go back and choose the top row to be P), or of opposite signs (as this would then contradict the stated triangular 4-cycle condition).

We can now embed into T_{2k} those conjugates of v_1, \dots, v_k that are present in H , by placing them in their appropriate columns on the bottom row of T_{2k} . Note that up to two triangular 4-cycles in H may become become parallelogram 4-cycles, and

vice versa, by this embedding. This induces an embedding G of H into T_{2k} , though without the signs of the edges yet agreeing. To achieve this agreement, we switch at these newly embedded vertices, if necessary, to ensure that all edges of negative slope have positive sign. We also switch at any (there can be no more than one) vertex in the bottom row that has no incident edge of negative slope, if necessary, to ensure that the incident edge of positive slope has negative sign.

We next claim that, after making these switchings, all edges of the embedding G do indeed have the same sign as the edges of T_{2k} . First consider an edge of G of positive slope. If not already made to have negative sign, such an edge must be part of a triangular 4-cycle where the two horizontal edges and the edge of negative slope all have positive sign. Hence, by the stated triangular 4-cycle condition, the edge of positive slope must have negative sign. (Note that because both the stated parallelogram 4-cycle condition and the triangular 4-cycle condition hold for H , the triangular 4-cycle condition holds for G .) Finally, every horizontal edge on the second row is part of an hourglass 4-cycle, which implies that it must have negative sign.

- (ii) Again take a maximum-length chordless path P in H , say of rank $k' \geq 5$. At least one of its endvertices must be charged, or else H could be embedded in T_{2k} , and by equivalence we may assume that if there are two charged vertices on P then they are not both negative. By switching we may assume that all edges of P are positive, so P can be embedded sign-consistently on the top row of one of C_{2k}^{++} or C_{2k}^{+-} , where $k = k'$ if P has two signed vertices, and $k = k' + 1$ if not. We then proceed as in (i), which ensures that all horizontal edges, and those of positive or negative slope, have the same sign as that of C_{2k}^{++} or C_{2k}^{+-} in the embedding. Finally, the triangle condition ensures that the vertical edges must have their signs as in C_{2k}^{++} or C_{2k}^{+-} .

□

A significant feature of the above result is that the stated parity conditions on the 4-cycles are all in some sense *local* conditions. The requirement that the rank should be at least 5 is best-possible: one can permute the vertices of T_8 so as to produce an isomorphic underlying graph but with the 4-cycle parity conditions failing to hold.

5. PROOF OF THEOREM 1

We now prove our main result, Theorem 1. We first quickly dispose of cases where at least one entry has modulus greater than 1 (Sections 5.1 and 5.2). The meat of the proof is concerned with adjacency matrices of charged signed graphs (Sections 5.3 and 5.4).

5.1. An entry with modulus at least 3. Let A be minimal noncyclotomic, and suppose that some entry $a = a_{ij}$ has modulus at least 3.

If this a diagonal entry ($i = j$), then (a) is a noncyclotomic submatrix, so must equal A . Since we can change the signs of all entries, we can assume that $a \geq 3$, and A is a member of our first infinite family in §6.1.

If $i \neq j$, then A has a noncyclotomic submatrix

$$\begin{pmatrix} b & a \\ a & c \end{pmatrix},$$

so this must equal A . Since A is minimal noncyclotomic, $|b| \leq 2$ and $|c| \leq 2$. By permuting and switching, we can suppose that $0 \leq |c| \leq b \leq 2$, and that if $b = c = 0$ then $a > 0$. We see that A is a member of one of the other infinite families in §6.1.

5.2. An entry with modulus 2. Now we suppose that A is minimal noncyclotomic, has all entries below 2 in modulus, with at least one entry a_{ij} having modulus equal to 2, and indeed working up to equivalence we may suppose that $a_{ij} = 2$.

If $i = j$, then since (2) is cyclotomic A has at least two rows, and contains a submatrix

$$\begin{pmatrix} 2 & a \\ a & b \end{pmatrix}.$$

We can choose this submatrix such that $a \neq 0$, or else A would be decomposable. No such 2-by-2 matrix is cyclotomic, so this must be the whole of A . Now $|b| \leq 2$ by hypothesis, hence A is a member of our second infinite family in §6.1.

If $i \neq j$, then A contains a submatrix

$$\begin{pmatrix} a & 2 \\ 2 & b \end{pmatrix}.$$

If either a or b is not zero, then this matrix is not cyclotomic, so equals A , which is seen to be equivalent to a member of the second or third infinite family in §6.1. Finally, if $a = b = 0$, then

$$\begin{pmatrix} a & 2 \\ 2 & b \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

is cyclotomic, so is not the whole of A . In this case, A has a submatrix equivalent to

$$\begin{pmatrix} 0 & 2 & a \\ 2 & 0 & b \\ a & b & c \end{pmatrix},$$

for some a, b, c (all of modulus at most 2, by hypothesis). Moreover since A is indecomposable we can choose our submatrix such that a and b are not both 0.

Note that $|c| < 2$, or else (using a, b not both 0) A would have a 2-by-2 noncyclotomic submatrix.

We quickly check that all cases are equivalent to one of the eight sporadic examples listed at the start of §6.2 that extend

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

5.3. Charged signed graphs: reduction to a finite search. From the classification of cyclotomic charged signed graphs recalled in Section 3, we know that every connected proper subgraph of a minimal noncyclotomic charged signed graph is either equivalent to a subgraph of one of the sporadic examples $S_7, S_8, S'_8, S_{14}, S_{16}$, or is equivalent to a subgraph of one of the three infinite families T_{2k} ($k \geq 3$), C_{2k}^{++} ($k \geq 2$), C_{2k}^{+-} ($k \geq 2$). We call a minimal noncyclotomic charged signed graph with n vertices *supersporadic* if it has a connected subgraph with $n - 1$ vertices that is equivalent to a subgraph of one of $S_7, S_8, S'_8, S_{14}, S_{16}$. It is clear that the number of supersporadic minimal noncyclotomic signed graphs is finite, and that they can all be computed (in principle) by running through every connected subgraph of $S_7, S_8, S'_8, S_{14}, S_{16}$ and considering all possible ways of adding a single vertex to the subgraph. In practice, this computation was performed rather more efficiently, as detailed in §5.4.

There remains the problem of finding any minimal noncyclotomic charged signed graphs that are not supersporadic. Potentially, this is an infinite search. We shall show, however, that the number of vertices in any such graph is at most 10, reducing the problem to a finite one.

Proposition 8. *Let G be a connected charged signed graph with $n \geq 11$ vertices, and such that every proper connected subgraph of G is equivalent to a subgraph of T_{2k}, C_{2k}^{++} or C_{2k}^{+-} , for some k . Then G is also equivalent to a subgraph of some T_{2k}, C_{2k}^{++} or C_{2k}^{+-} .*

It follows straight from this result that a minimal noncyclotomic charged signed graph that is not supersporadic can have no more than 10 vertices.

Proof. Take such a G . Our first aim is to show that G has a profile. Take a chordless path or cycle P with a maximal number of vertices, and let x and y be the endvertices of P if P is a path, and let them be any two adjacent vertices of P if P is a cycle. If the maximal number of vertices for a chordless cycle equals that for a chordless path, then take P to be a path.

First note that no vertex of G is adjacent to x but to no other vertex on P , or else we could either grow P to a longer chordless path, or replace a chordless cycle P by a chordless path of equal length. Similarly for y . It follows that $G - \{x\}$ is connected, and since it contains at least 10 vertices it has rank at least 5, and hence P contains at least 5 vertices. In the case where P is a path, if there were a vertex not on P adjacent to both x and y but to no other vertex in P , then P could be extended to a longer chordless cycle: a contradiction. In the case where P is a cycle $xyv_1v_2\cdots$, if there were a vertex z not on P adjacent to both x and y but to no other vertex on P , then $P \cup \{z\} - \{v_2\}$ would be a proper connected subgraph of G containing the triangle zyx , hence containing two charged vertices, yet not equivalent to a subgraph of C_{2k}^{++} or C_{2k}^{+-} (here using that $P - \{v_2\}$ contains at least 4 vertices): another contradiction. In all cases we see that G has no vertex adjacent to either x or y that is not also adjacent to some vertex on P , so $G - \{x, y\}$ is connected.

Now $G - \{x, y\}$ has rank say r at least 5 so that, by Lemma 6, it has a uniquely determined profile. As the profiles of $G - \{x\}$ and $G - \{y\}$ are also uniquely determined,

they can each be obtained by adding y or x to the profile of $G - \{x, y\}$. Leaving aside for the moment the issue of whether or not x and y are adjacent in G , we see that all other possible adjacencies of x in G can be read off from the profile of $G - \{y\}$, and all other possible adjacencies of y in G can be read off from the profile of $G - \{x\}$. Thus we can merge the profiles of $G - \{x\}$ and $G - \{y\}$ to obtain a new sequence of ‘columns’, \mathcal{C} , which we shall show is in fact the profile of G . In this merging, certainly columns $2, 3, \dots, r-1$ carry over unchanged. The vertices x and y will lie either in columns 1 or r or in a new singleton column to the left or right of the profile of $G - \{x, y\}$. As x and y are the endpoints of a maximal chordless path or cycle, they must be in opposite end columns.

First suppose that at least one of the vertices in the column of x is adjacent to at least one of the vertices in the column of y . Then by deleting column 3 of $G - \{x, y\}$ we obtain another connected proper subgraph of G , which is therefore equivalent to a subgraph of some T_{2k} , C_{2k}^{++} or C_{2k}^{+-} . Furthermore, it is determined by its profile, so that in fact every vertex in the column of x is adjacent to every vertex in the column of y . Thus \mathcal{C} is indeed the profile of G . The local conditions of Proposition 7 are seen to hold, and G is equivalent to a subgraph of some T_{2k} .

Alternatively, it may be that no vertex in the column of x is adjacent to any other vertex in the column of y . Then again \mathcal{C} is the profile of G . The local conditions of Proposition 7 (perhaps this time with charged vertices at one or both ends) hold, since they hold for both $G - \{x\}$ and $G - \{y\}$, so that G is equivalent to a subgraph of some T_{2k} , C_{2k}^{++} or C_{2k}^{+-} . \square

5.4. Charged signed graphs: details of the finite search. By the *degree* of an uncharged vertex v we mean the number of adjacent vertices (those vertices w such that there is an edge of either sign between v and w). We define the degree of a charged vertex to be one more than the number of adjacent vertices.

Lemma 9. *Up to equivalence, the graphs 5b, 5y and 6c are the only minimal noncyclotomic charged signed graphs containing a vertex of degree greater than 4.*

Proof. All such graphs G with up to 6 vertices are readily found by searching. Now suppose that G has at least 7 vertices, with a vertex x of degree at least 5. Suppose first that x is uncharged. Take x_1, \dots, x_5 to be five of the neighbours of x . Since G has at least 7 vertices, the subgraph induced by x, x_1, \dots, x_5 must be cyclotomic. Similarly if x is charged, the subgraph induced by x and four of its neighbours must be cyclotomic. Yet no cyclotomic charged signed graph has a vertex of degree greater than 4. \square

We generated lists of connected charged signed graphs with small numbers of vertices, and either cyclotomic or minimal noncyclotomic, starting with the list of the two inequivalent 1-vertex charged signed graphs (both of these are cyclotomic). Having produced a list of r -vertex charged signed graphs, to compute a list of $r+1$ -vertex charged signed graphs we considered all ways of adding a vertex to all the r -vertex cyclotomics; this would either give a cyclotomic (which was stored in the new list), or a noncyclotomic, and in the latter case minimality was tested, and minimal noncyclotomics added to the new list.

Since we were working up to equivalence, we deleted any graphs from our lists that were quickly found to be equivalent to another one. The quick test that we used did not always identify equivalent graphs. Repeats of cyclotomic graphs were tolerated; when producing final lists of minimal noncyclotomic graphs, possible repeats were investigated by hand.

The search went exhaustively up to 15-vertex charged signed graphs, making use of Lemma 9 once the number of vertices was large enough. The connected minimal noncyclotomics found were precisely those displayed in Section 6 below. After Proposition 8, we then needed only to consider supergraphs of subgraphs of S_{16} , where the subgraph contained at least 15 vertices, but we found no more minimal noncyclotomics.

One consequence of this search is that it reveals all those minimal noncyclotomics that are not supersporadic: $3a, 3f, 4d, 4f, 4n, 5f, 5y, 6k, 6l$. In particular, the ‘11’ in Proposition 8 can be reduced to ‘7’.

6. THE MINIMAL INDECOMPOSABLE NONCYCLOTOMIC MATRICES

6.1. The infinite families. For any natural number n greater than 2,

$$(n)$$

is minimal noncyclotomic. Its single eigenvalue is $n \geq 2$, and its Mahler measure is $(n + \sqrt{n^2 - 4})/2 \geq 2.618$.

The following are infinite families of minimal noncyclotomic matrices:

$$\begin{aligned} \begin{pmatrix} 2 & a \\ a & b \end{pmatrix} & \quad |a| \geq 1, |b| \leq 2, \\ \begin{pmatrix} 1 & a \\ a & b \end{pmatrix} & \quad |a| \geq 2, |b| \leq 1, \\ \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} & \quad a \geq 3. \end{aligned}$$

One readily checks that in all cases the spectral radius is at least $\sqrt{2}$ and the Mahler measure is greater than 1.722.

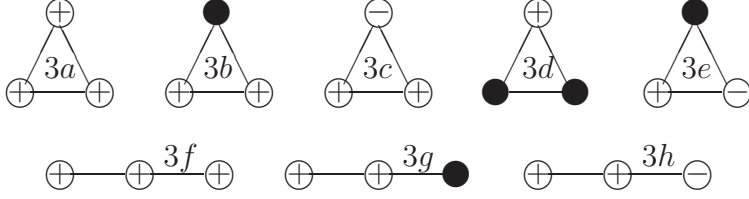
6.2. The sporadic examples. First we list representatives of the equivalence classes of matrices that extend $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$:

$$\begin{aligned} & \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

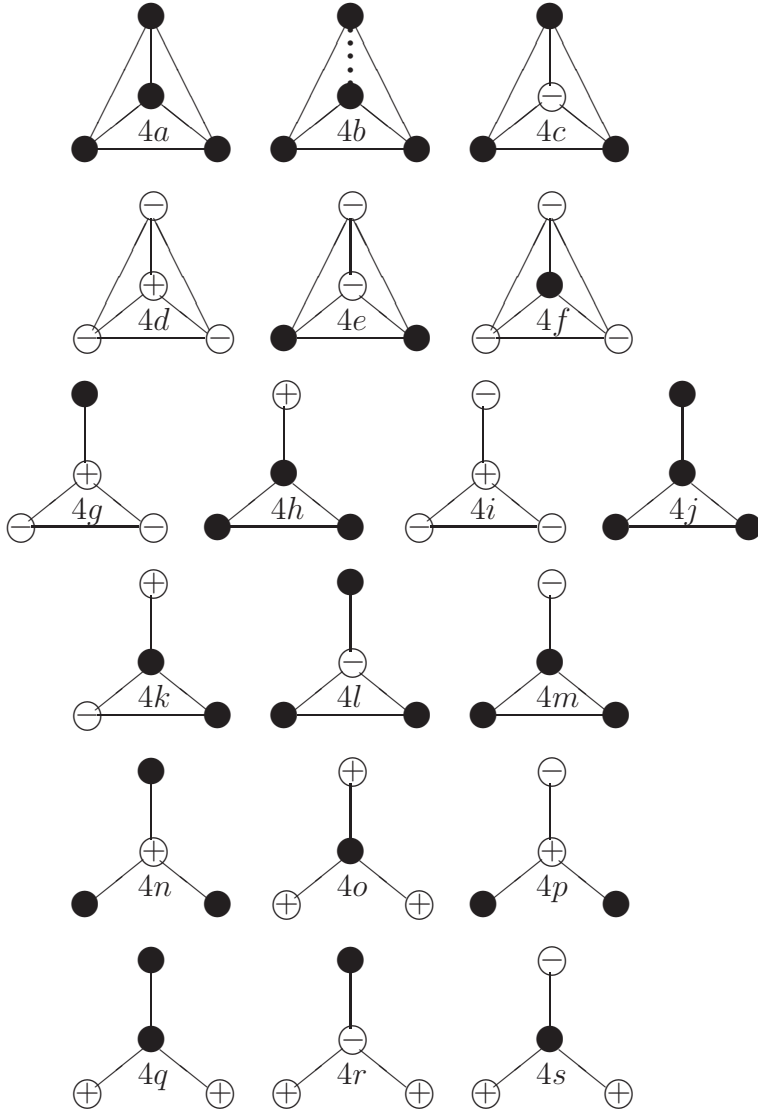
In each case, the spectral radius is at least $\sqrt{5}$ and the Mahler measure is greater than 2.081.

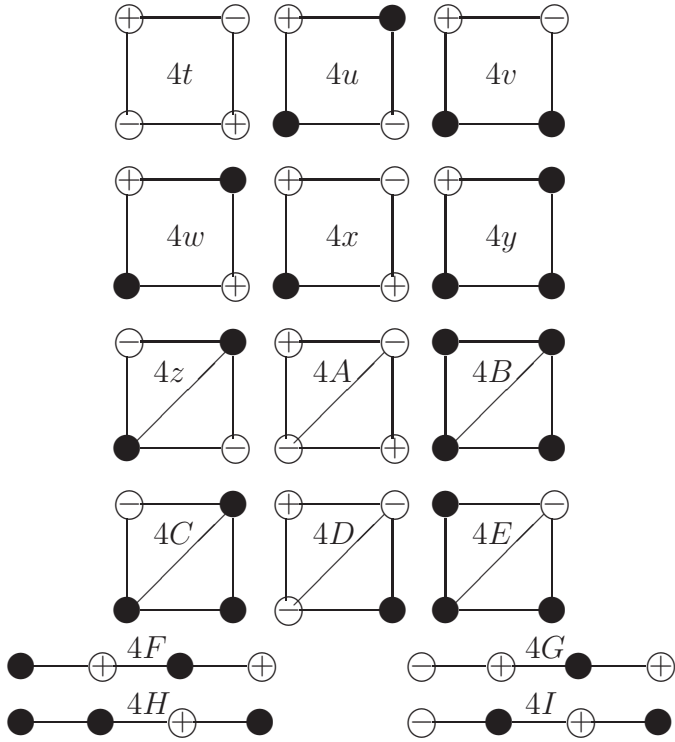
The remaining sporadic examples are all adjacency matrices of charged signed graphs. We display them in order of their number of vertices. Then in Table 6.2 we give their spectral radius and Mahler measure.

3-vertex minimal noncyclotomic charged signed graphs:

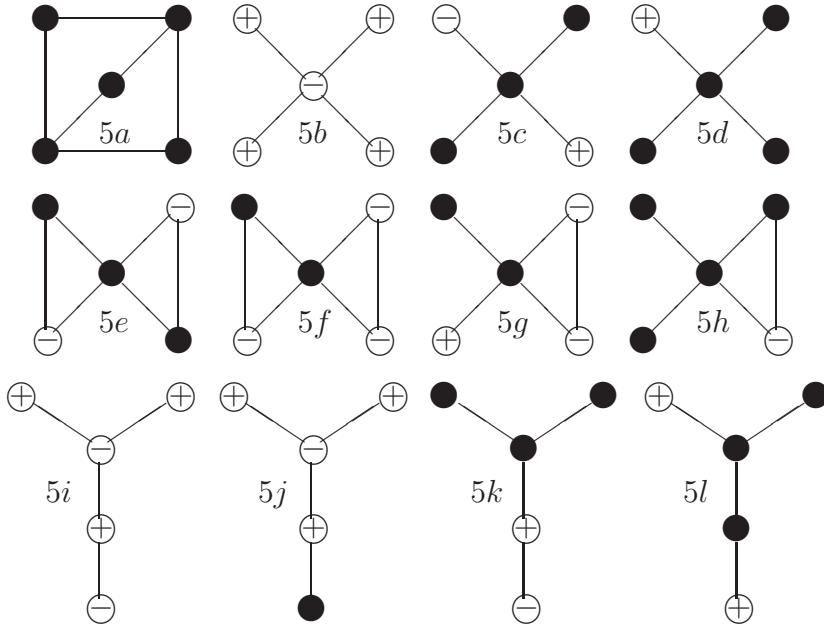


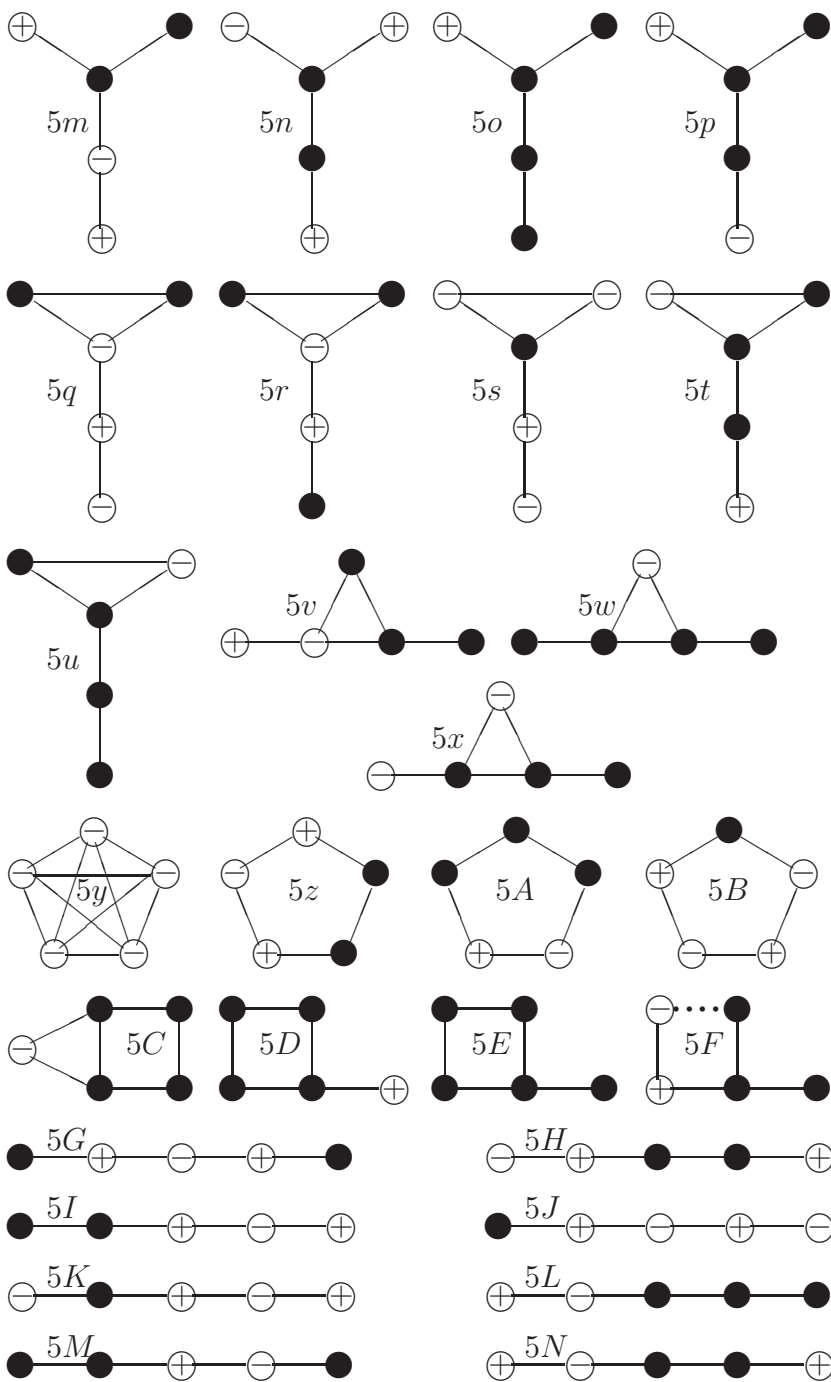
4-vertex minimal noncyclotomic charged signed graphs:



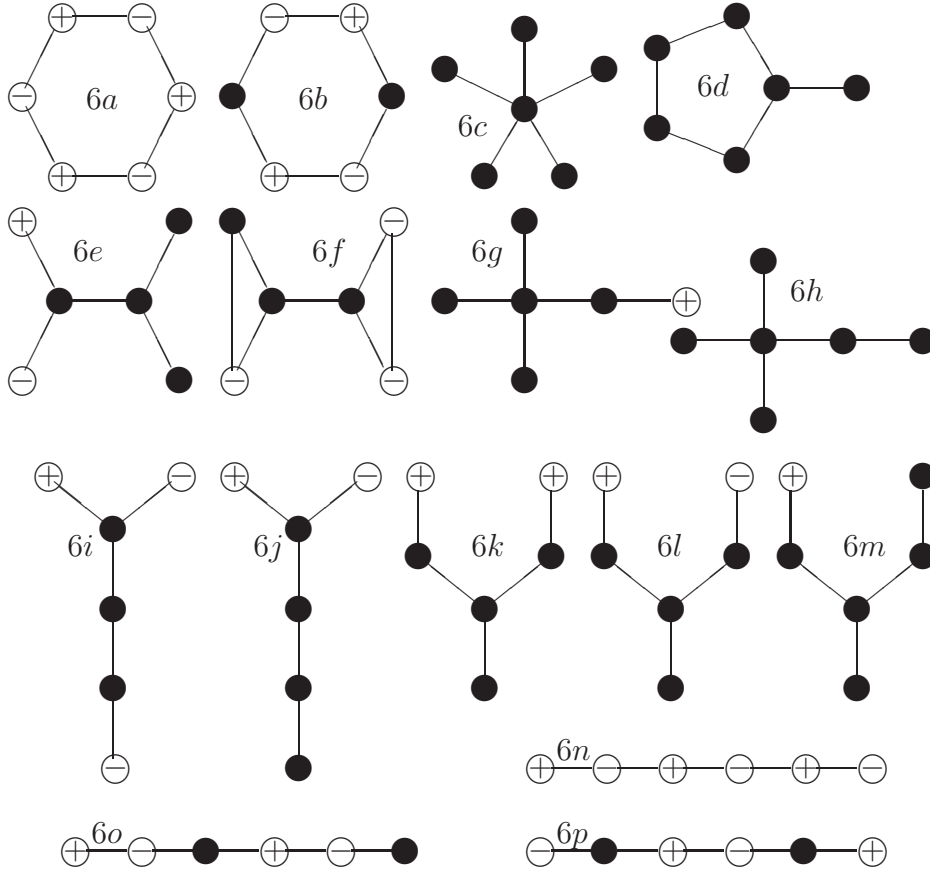


5-vertex minimal noncyclotomic charged signed graphs:

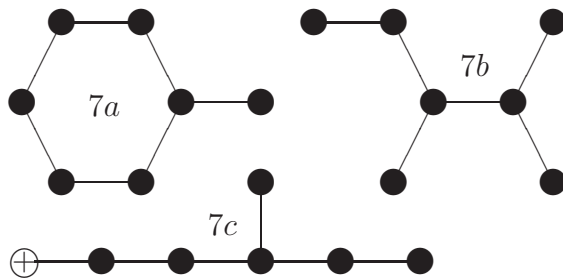




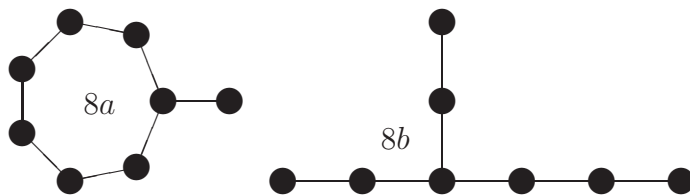
6-vertex minimal noncyclotomic charged signed graphs:

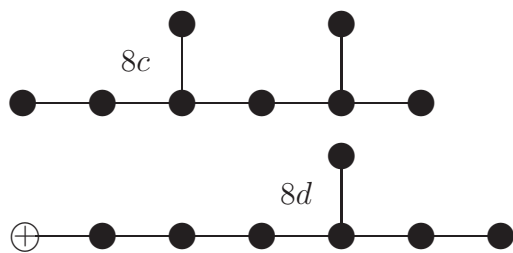


7-vertex minimal noncyclotomic charged signed graphs:

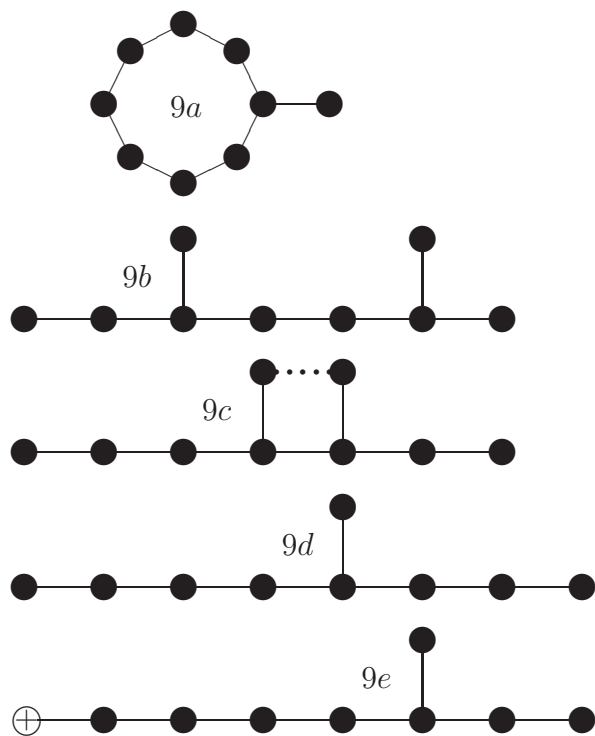


8-vertex minimal noncyclotomic charged signed graphs:

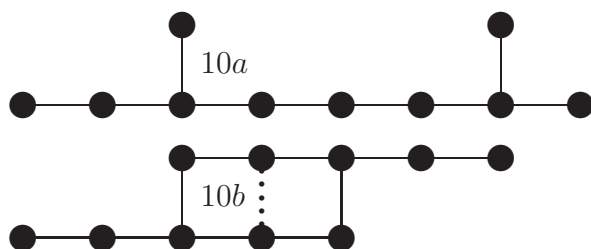




9-vertex minimal noncyclotomic charged signed graphs:



10-vertex minimal noncyclotomic charged signed graphs:



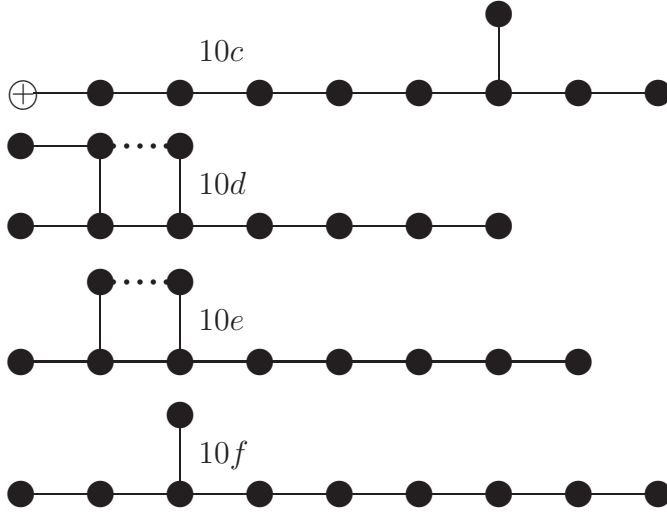


Table 3: Spectral radius and Mahler measure of the minimal noncyclotomic charged signed graphs. The ‘outside’ column gives the number of eigenvalues outside the interval $[-2, 2]$.

#	Name	Spectral radius	Outside	Mahler measure
1	3a	3.00000	1	2.61803
2	3b	2.73205	1	2.29663
3	3c	2.56155	1	2.08102
4	3d	2.41421	1	1.88320
5	3e	2.21432	1	1.58235
6	3f	2.41421	1	1.88320
7	3g	2.24698	1	1.63557
8	3h	2.17009	1	1.50614
9	4a	3.00000	1	2.61803
10	4b	2.23607	2	2.61803
11	4c	2.79129	1	2.36921
12	4d	2.73205	1	2.29663
13	4e	2.56155	1	2.08102
14	4f	2.30278	1	1.72208
15	4g	2.30278	1	1.72208
16	4h	2.30278	1	1.72208
17	4i	2.21432	1	1.58235
18	4j	2.17009	1	1.50614
19	4k	2.14386	1	1.45799
20	4l	2.11491	1	1.40127
21	4m	2.11491	1	1.40127

Table 3: (continued) Spectral radius and Mahler measure of the minimal noncyclotomic charged signed graphs.

#	Name	Spectral radius	Outside	Mahler measure
22	$4n$	2.30278	1	1.72208
23	$4o$	2.30278	1	1.72208
24	$4p$	2.21432	1	1.58235
25	$4q$	2.17009	1	1.50614
26	$4r$	2.11491	1	1.40127
27	$4s$	2.11491	1	1.40127
28	$4t$	2.23607	2	2.61803
29	$4u$	2.23607	2	2.61803
30	$4v$	2.18890	2	2.36921
31	$4w$	2.56155	1	2.08102
32	$4x$	2.41421	1	1.88320
33	$4y$	2.34292	1	1.78164
34	$4z$	2.23607	2	2.61803
35	$4A$	2.56155	1	2.08102
36	$4B$	2.56155	1	2.08102
37	$4C$	2.41421	1	1.88320
38	$4D$	2.34292	1	1.78164
39	$4E$	2.30278	1	1.72208
40	$4F$	2.19353	1	1.54720
41	$4G$	2.12676	1	1.42501
42	$4H$	2.09529	1	1.36000
43	$4I$	2.06150	1	1.28064
44	$5a$	2.44949	2	3.73205
45	$5b$	2.23607	2	2.61803
46	$5c$	2.13578	2	2.08102
47	$5d$	2.19869	1	1.55603
48	$5e$	2.30278	1	1.72208
49	$5f$	2.17009	1	1.50614
50	$5g$	2.19869	1	1.55603
51	$5h$	2.17009	1	1.50614
52	$5i$	2.10100	2	1.88320
53	$5j$	2.15976	2	1.84752
54	$5k$	2.14386	1	1.45799
55	$5l$	2.13883	1	1.44842
56	$5m$	2.09529	1	1.36000
57	$5n$	2.09118	1	1.35098
58	$5o$	2.06150	1	1.28064
59	$5p$	2.03850	1	1.21639
60	$5q$	2.10100	2	1.88320

Table 3: (continued) Spectral radius and Mahler measure of the minimal noncyclotomic charged signed graphs.

#	Name	Spectral radius	Outside	Mahler measure
61	$5r$	2.15976	2	1.84752
62	$5s$	2.14386	1	1.45799
63	$5t$	2.11491	1	1.40127
64	$5u$	2.02642	1	1.17628
65	$5v$	2.13797	2	1.83505
66	$5w$	2.11491	1	1.40127
67	$5x$	2.06150	1	1.28064
68	$5y$	3.00000	1	2.61803
69	$5z$	2.34292	1	1.78164
70	$5A$	2.17009	1	1.50614
71	$5B$	2.22833	1	1.60545
72	$5C$	2.34292	1	1.78164
73	$5D$	2.25619	2	2.22371
74	$5E$	2.13578	2	2.08102
75	$5F$	2.05411	1	1.26123
76	$5G$	2.11491	1	1.40127
77	$5H$	2.10637	1	1.38364
78	$5I$	2.08508	1	1.33731
79	$5J$	2.06659	1	1.29349
80	$5K$	2.05411	1	1.26123
81	$5L$	2.04314	1	1.23039
82	$5M$	2.03850	1	1.21639
83	$5N$	2.02642	1	1.17628
84	$6a$	2.23607	2	2.61803
85	$6b$	2.21432	2	2.50382
86	$6c$	2.23607	2	2.61803
87	$6d$	2.11491	1	1.40127
88	$6e$	2.10100	2	1.88320
89	$6f$	2.12676	1	1.42501
90	$6g$	2.15976	2	1.84752
91	$6h$	2.07431	2	1.72208
92	$6i$	2.07852	2	1.50646
93	$6j$	2.02852	2	1.40127
94	$6k$	2.11491	1	1.40127
95	$6l$	2.02852	2	1.40127
96	$6m$	2.04671	1	1.24073
97	$6n$	2.06082	2	1.63557
98	$6o$	2.07103	2	1.57837
99	$6p$	2.04907	2	1.55603

Table 3: (continued) Spectral radius and Mahler measure of the minimal noncyclotomic charged signed graphs.

#	Name	Spectral radius	Outside	Mahler measure
100	7a	2.10100	2	1.88320
101	7b	2.05288	2	1.58235
102	7c	2.03850	1	1.21639
103	8a	2.09118	1	1.35098
104	8b	2.02852	2	1.40127
105	8c	2.04208	2	1.50614
106	8d	2.03334	1	1.20003
107	9a	2.08397	2	1.78164
108	9b	2.03565	2	1.45799
109	9c	2.02368	2	1.36000
110	9d	2.01532	2	1.28064
111	9e	2.02986	1	1.18837
112	10a	2.03144	2	1.42501
113	10b	2.02642	2	1.38364
114	10c	2.02739	2	1.25364
115	10d	2.01348	2	1.26123
116	10e	2.00960	2	1.21639
117	10f	2.00659	2	1.17628

7. CHARGED SIGNED GRAPHS OF SMALL SPECTRAL RADIUS

Any noncyclotomic charged signed graph of spectral radius less than 2.019 must, by interlacing, contain as a subgraph a minimal noncyclotomic charged signed graph of spectral radius less than 2.019. Thus the former can be ‘grown’ from the latter by successively adding a vertex of charge -1 , 0 or 1 , and adjoining it in all possible ways to the vertices of the current graph. Furthermore, we claim that we can assume that the vertex adjoined is of degree at most 4.

For suppose that the added vertex, v say, is of degree at least 5 and that the resulting graph G is of spectral radius less than 2.019. Consider the two cases

(i) v charged.

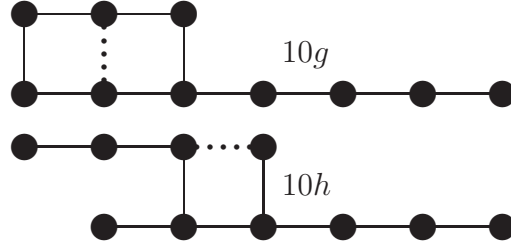
Consider the subgraph G_5 of G on v and four of its neighbours. As no maximal cyclotomic graph contains a charged vertex of degree 5, G_5 cannot be cyclotomic. It therefore contains a minimal noncyclotomic subgraph with at most 5 vertices and spectral radius less than 2.019. However, from our results, there is no minimal noncyclotomic graph of spectral radius less than 2.019 with fewer than 9 vertices.

(ii) v neutral.

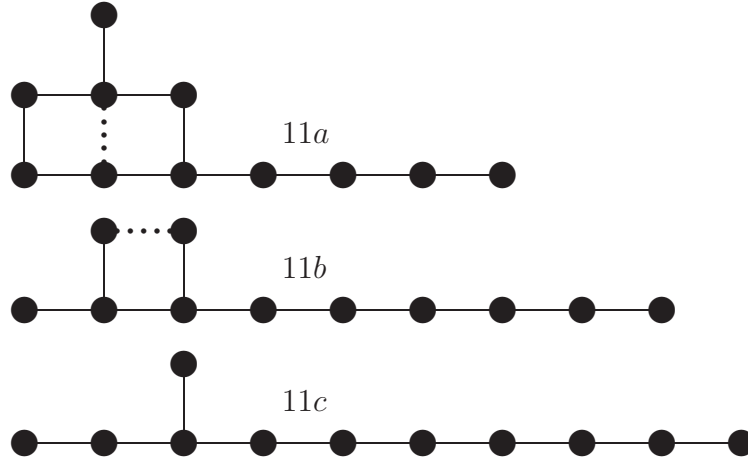
Consider the subgraph G_6 of G on v and five of its neighbours. As no maximal cyclotomic graph contains a vertex of degree 5, G_6 cannot be cyclotomic. It therefore contains a minimal noncyclotomic subgraph with at most 6 vertices, and we obtain the same contradiction.

The results of this growing procedure are shown in the pictures below. Together with the minimal examples of small spectral radius, $9d$, $10d$, $10e$, $10f$, we produce Table 1, and establish Theorem 3. It turns out that all these charged signed graphs are in fact simply signed graphs. Several are starlike trees $T_{a,b,c}$, as in Figure 1.

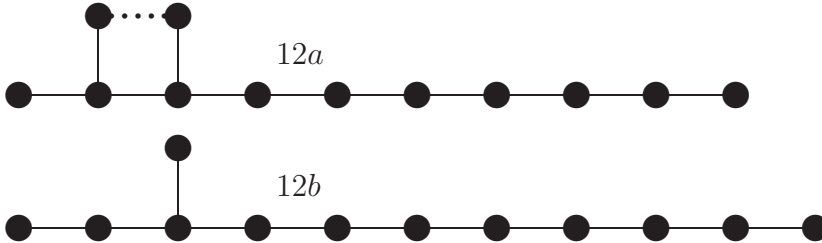
Connected 10-vertex nonminimal noncyclotomic (charged) signed graphs with spectral radius < 2.019 :



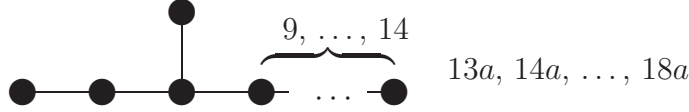
Connected 11-vertex nonminimal noncyclotomic (charged) signed graphs with spectral radius < 2.019 :



Connected 12-vertex nonminimal noncyclotomic (charged) signed graphs with spectral radius < 2.019 :



Connected 13- to 18-vertex nonminimal noncyclotomic (charged) signed graphs with spectral radius < 2.019 :



8. CHARGED SIGNED GRAPHS OF SMALL MAHLER MEASURE

Any noncyclotomic charged signed graph of Mahler measure less than 1.3 must, by interlacing, contain as a subgraph a minimal noncyclotomic charged signed graph of Mahler measure less than 1.3. Thus the former can be grown from the latter. Again we claim that we can assume that the vertex adjoined is of degree at most 4.

For suppose that v is of degree at least 5 and that the resulting graph G is of Mahler measure less than 1.3. Again, consider the two cases

(i) v charged.

Consider the subgraph G_5 of G on v and four of its neighbours. As no maximal cyclotomic graph contains a charged vertex of degree 5, G_5 cannot be cyclotomic. It therefore contains a minimal noncyclotomic subgraph. However, from our results, $5b$ and $5y$ are the only minimal noncyclotomic graphs containing a vertex of degree 5, and their Mahler measures are all greater than 1.3. Hence G_5 is not minimal noncyclotomic. It therefore contains a minimal noncyclotomic subgraph with at most four vertices, and Mahler measure less than 1.3. From our results, the only one is $4I$, which, having no vertices of degree 4, must be the subgraph $G_5 - \{v\}$. We now check by computer that when v is adjoined to the four vertices of $4I$ with all 16 possible choices of edge signs then in each case the resulting graph has Mahler measure greater than 1.3.

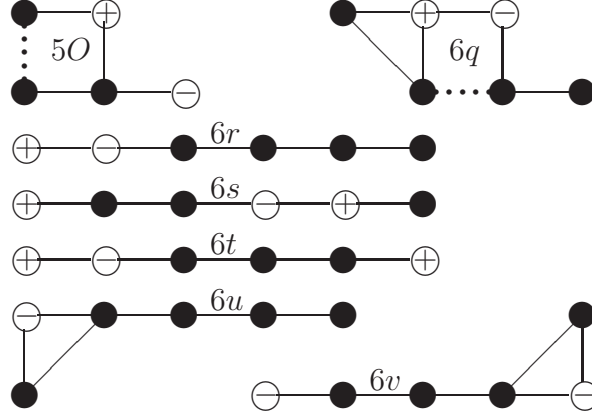
(ii) v neutral.

Consider the subgraph G_6 of G on v and five of its neighbours. Again, G_6 cannot be cyclotomic, and so contains a minimal noncyclotomic subgraph. Now G_6 itself is not minimal, as no 6-vertex minimal graph of Mahler measure less than 1.3 contains a vertex of degree greater than 3. Suppose that G_6 has a minimal noncyclotomic subgraph containing v . It can then have at most four vertices, so must be $4I$. But $4I$ has no neutral vertex of degree 3. Hence no minimal noncyclotomic subgraph of G_6 contains v . We now check by computer that when v is adjoined to $4I$ as above (and also to another vertex if necessary, which might itself be adjacent to vertices of $4I$), or to the five vertices of the ten minimal 5-vertex noncyclotomic subgraphs of Mahler measure less than 1.3 with all 2^5 possible choices of edge signs, then in each case the resulting graph has Mahler measure greater than 1.3.

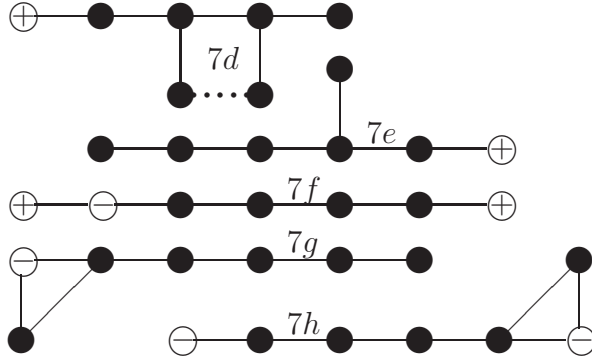
The new charged signed graphs found by this growing procedure are shown below. Together with the minimal examples of small Mahler measure, $4I$, $5o$, $5p$, $5u$, $5x$, $5F$, $5J$, $5K$, $5L$, $5M$, $5N$, $6m$, $7c$, $8d$, $9d$, $9e$, $10c$, $10d$, $10e$, $10f$, and the nonminimal examples seen when considering small spectral radius, $10g$, $10h$, $11a$, $11b$, $11c$, $12b$, $13a$, $14a$, we

produce Table 2, and establish Theorem 4. All the new examples include at least one charged vertex.

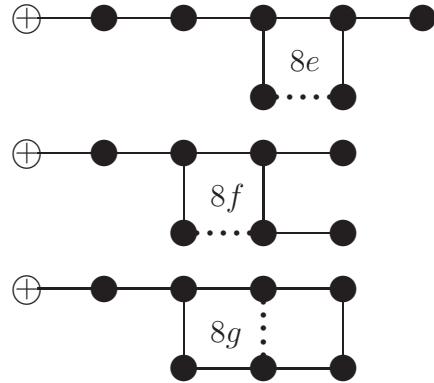
Connected 5- and 6-vertex nonminimal noncyclotomic charged signed graphs with Mahler measure < 1.3 :

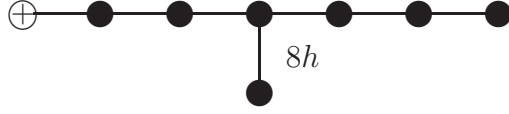


Connected 7-vertex nonminimal noncyclotomic charged signed graphs with Mahler measure < 1.3 :

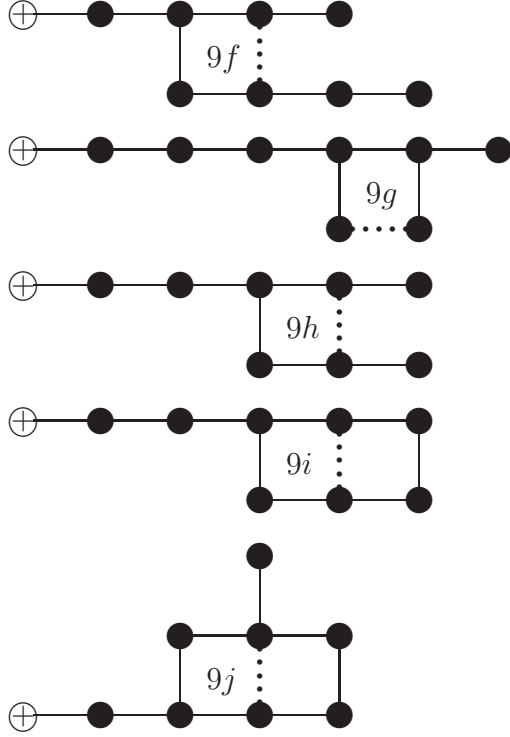


Connected 8-vertex nonminimal noncyclotomic charged signed graphs with Mahler measure < 1.3 :

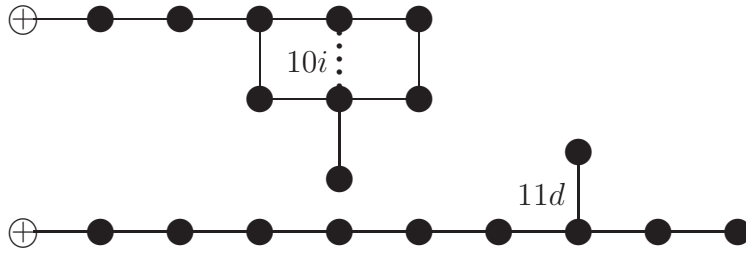




Connected 9-vertex nonminimal noncyclotomic charged signed graphs with Mahler measure < 1.3 :



Connected 10- and 11-vertex nonminimal noncyclotomic charged signed graphs with Mahler measure < 1.3 other than 10g, 10h, 11a, 11b, 11c:



Acknowledgement. We thank Klas Markström for pointing us to the reference [12].

REFERENCES

- [1] R. Bhatia, Matrix analysis. New York, NY: Springer 1996.
- [2] P. Borwein, E. Dobrowolski, M. Mossinghoff, Lehmer's problem for polynomials with odd coefficients. *Ann. of Math.* **166** (2007), 347–366.
- [3] D.W. Boyd, Small Salem numbers. *Duke Math. J.* **44** (1977), 315–328.
- [4] A.E. Brouwer and A. Neumaier, The graphs with spectral radius between 2 and $\sqrt{2 + \sqrt{5}}$. *Linear Algebra Appl.* **114/115** (1989), 273–276.
- [5] A.L. Cauchy, Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planètes, Oeuvres complètes, IIIème Série, **9**, Gauthier-Villars, 174–195 (1829).
- [6] P.J. Cameron, J.J. Seidel and S.V. Tsaranov, Signed graphs, root lattices, and Coxeter groups, *J. Algebra* **164** (1994), 173–209.
- [7] D. Cvetković, M. Doob and I. Gutman, On graphs whose eigenvalues do not exceed $\sqrt{2 + \sqrt{5}}$, *Ars Combinatoria* **14** (1982), 225–239.
- [8] D. Cvetković and P. Rowlinson, The largest eigenvalue of a graph: a survey. *Linear and Multilinear Algebra* **28** (1990), 3–33.
- [9] E. Dobrowolski, A note on integer symmetric matrices and Mahler's measure, *Canad. Math. Bull.* **51** (2008), 57–59.
- [10] D.R. Estes and R.M. Guralnick, Minimal polynomials of integral symmetric matrices, *Linear Algebra Appl.* **192** (1993), 83–99.
- [11] S. Fisk, A very short proof of Cauchy's interlace theorem for eigenvalues of Hermitian matrices, *Amer. Math. Monthly* **112** (2005), 118. See also [arXiv:math.CA/0502408v1](https://arxiv.org/abs/math.CA/0502408v1).
- [12] R. Hayward, B.A. Reed, Forbidding holes and antiholes. in J.L. Ramrez Alfonsn, (ed.) et al., Perfect graphs. Chichester: Wiley. 113–137 (2001).
- [13] L. Kronecker, Zwei sätze über gleichungen mit ganzzahligen coefficienten, *J. Reine Angew. Math.* **53** (1857), 173–175.
- [14] J.F. McKee and C.J. Smyth, Salem numbers, Pisot numbers, Mahler measure, and graphs, *Experimental Mathematics* **14** (2005), 211–229.
- [15] J.F. McKee and C.J. Smyth, Integer symmetric matrices having all their eigenvalues in the interval $[-2, 2]$, *J. Algebra* **317** (2007), 260–290.
- [16] M. Mossinghoff, List of small Salem numbers,
<http://www.cecm.sfu.ca/~mjm/Lehmer/lists/SalemList.html>
- [17] J.B. Shearer, On the distribution of the maximum eigenvalue of graphs, *Linear Algebra Appl.* **114/115** (1989), 17–20.
- [18] C.J. Smyth, The Mahler measure of algebraic numbers: a survey. *Number theory and polynomials*, pp. 322–349. (Conference proceedings, University of Bristol, 3–7 April 2006, editors James McKee and Chris Smyth). LMS Lecture Note Series **352**, Cambridge University Press, Cambridge, 2008. See also [arXiv:math.NT/0701397v2](https://arxiv.org/abs/math.NT/0701397v2).

DEPARTMENT OF MATHEMATICS, ROYAL HOLLOWAY, UNIVERSITY OF LONDON, EGHAM HILL, EGHAM, SURREY TW20 0EX, UK

E-mail address: James.McKee@rhul.ac.uk

SCHOOL OF MATHEMATICS AND MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES, UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, KING'S BUILDINGS, MAYFIELD ROAD, EDINBURGH EH9 3JZ, SCOTLAND, U.K.

E-mail address: C.Smyth@ed.ac.uk